Eur. Phys. J. B **60**, 67–73 (2007)
DOI: 10.1140/epjb/e2007-00328-7

THE EUROPEAN PHYSICAL JOURNAL B

Decomposition of the Fock space in two-dimensional triangle and honeycomb lattice systems

B. Kim and M.-H. Chung[a]

Department of Physics, Hong-Ik University, Chochiwon, Choongnam 339-701, Korea

Received 30 July 2007 / Received in final form 13 October 2007
Published online 29 November 2007 –

Abstract. We consider the symmetry group inherent in two-dimensional triangle and honeycomb lattice systems. We find analytically and numerically the character of the reducible representation for the corresponding Fock space. Using the irreducible characters and the reducible character of the representation, we decompose the Fock space explicitly. For example, we calculate the multiplicity of each irreducible representation contained in the Fock space.

PACS. 02.20.-a Group theory – 71.27.+a Strongly correlated electron systems; heavy fermions

1 Introduction

Many numerical methods have been used in order to understand strongly correlated electron systems. Quantum Monte Carlo [1], exact diagonalization [2], and density matrix renormalization group [3] are main methods deployed recently. In the method of exact diagonalization, the key issue is how to handle symmetries embedded in systems. The reason for analyzing symmetries is that using them we find not only quantum numbers of the systems, but also we can block-diagonalize the Hamiltonian matrix. The best way to study symmetries would be to apply group representation theory to the Hilbert space.

There are concrete mathematical studies of group theory for translation, rotation, and reflection symmetries. The corresponding symmetry groups are called *plane group* [4] and *space group* [5] for two and three-dimensional lattice systems, respectively. It is known that plane group is classified into 17 different groups, and there are 230 space groups [6].

In group representation theory [7], the main concern is to find all irreducible components. Sometimes it is meaningful to calculate a reducible representation as far as the corresponding space is physically important. The Fock space is a basic Hilbert space for strongly correlated systems, and therefore, it is worthwhile to consider the reducible representation for the Fock space.

The authors considered a two-dimensional square lattice system without focusing on a specific model in the previous paper [8]. The corresponding symmetry group is p4mm in the crystallographic notation [5]. Using irreducible representations of the group and the reducible representation of the Fock space, the Fock space is decomposed into the irreducible components. As a result, we found the dimensions of the Hamiltonian submatrix, which correspond to multiplicities.

Beyond the square lattice system, it is necessary to analyze the triangle lattice systems in order to study the nature of the Wigner crystal [9]. Recently the seminal experiment [10] attracts many interests on graphene, which is one of honeycomb lattice systems. It is essential to investigate the plane group embedded in the triangle and honeycomb lattice systems.

In this paper, we present a thorough procedure of investigating the triangle and honeycomb lattice systems. Doing so, we show the relation between the triangle lattice and the honeycomb lattice.

This paper is organized as follows. We introduce the symmetry group, p6mm, involved in two-dimensional triangle lattice. We also find that two-dimensional honeycomb lattice system has the same symmetry group as that of triangle. We present the irreducible characters of the group in Section 3, and we find the reducible character of the Fock space in Section 4. In Section 5, for example, we calculate the multiplicities of the irreducible representations of the Fock space in two cases of triangle and honeycomb lattices. We make a conclusion in Section 6.

2 Symmetries in triangle and honeycomb lattice systems

A two-dimensional lattice system has several symmetries, which are translation, rotation and reflection. In order to understand the symmetries in lattice systems, we present lattice points using integer multiples of the two unit basis vectors.

[a] e-mail: mhchung@hongik.ac.kr

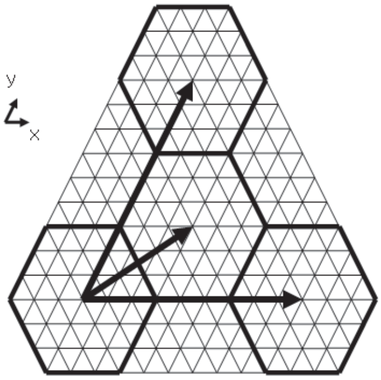

Fig. 1. The unit basis vectors x and y are presented in a triangle lattice system. The bold lines represent the boundary for the periodicity of $N = 3$. It is shown that the periodicity along the direction of $x + y$ is N, while the periodicity along the direction of x and y is $3N$.

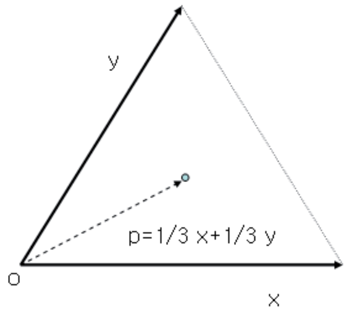

Fig. 2. In addition to the basis vectors x and y from the origin o, the center of the triangle is represented by p.

2.1 Triangle

We begin by considering the triangle lattice system first. We use two unit vectors x and y to present lattice points. As shown in Figure 1, we use y, which is tilted by 60° with respect to x. Now each position is represented by a vector $ix + jy$ with two integers i and j.

Let us denote T_x and T_y for the translation by one lattice site in the x and y direction, respectively. For 60° counterclockwise rotation about the origin, we use C_6. For the reflections about the six axes, we use σ_x, σ_{x+y}, σ_y, σ_{-x+2y}, σ_{-x+y}, σ_{-2x+y}.

Explicitly these operators act on sites as follows:

$$T_x(ix + jy) = (i + 1)x + jy, \tag{1}$$
$$T_y(ix + jy) = ix + (j + 1)y, \tag{2}$$
$$C_6(ix + jy) = -jx + (i + j)y, \tag{3}$$
$$\sigma_x(ix + jy) = (i + j)x - jy. \tag{4}$$

We note that the other reflection operators are written in terms of C_6 and σ_x as

$$\sigma_{x+y} = C_6\sigma_x, \tag{5}$$
$$\sigma_y = C_6^2\sigma_x, \tag{6}$$
$$\sigma_{-x+2y} = C_6^3\sigma_x, \tag{7}$$
$$\sigma_{-x+y} = C_6^4\sigma_x, \tag{8}$$
$$\sigma_{-2x+y} = C_6^5\sigma_x. \tag{9}$$

In order to make the system finite, we impose the boundary condition, with which T_x and T_y satisfy

$$T_x^N T_y^N = 1, \text{ and } T_x^{3N} = T_y^{3N} = 1, \tag{10}$$

as explained in Figure 1. Here we define T_t

$$T_t \equiv T_x T_y, \text{ and } T_t^N = 1. \tag{11}$$

In this boundary condition, it is important to note that the number of lattice sites is given by $3N^2$. To see this, we note that $1 + 1 \times 6 + 2 \times 6 + \cdots + (N - 1) \times 6$ for inside, $(N - 1) \times 6/2$ for six border lines, and $6/3$ for six corner points of the boundary. Thus, we find that $1 + (N - 1)N \times 3 + (N - 1) \times 3 + 2 = 3N^2$. If we consider the system of M fermions, the dimension of the corresponding Fock space is given by ${}_{3N^2}C_M$, where $3N^2$ is the number of sites.

We note that rotation and reflection symmetries around any point $p = ax + by$ can be represented by the combination of the rotation and reflection about the origin and translations. Let O be an operator representing a rotation or a reflection about the origin. Then any such operator about p can be written as $T_p O T_p^{-1} = T_{p'} O$ where $p' = T_p O(-p)$. For example, let us choose p as a center of some triangle, say, $p = \frac{1}{3}x + \frac{1}{3}y$ as shown in Figure 2. Then, the rotation by 60° about p is written as $T_p C_6 T_p^{-1} = T_{p'} C_6$ with $p' = \frac{2}{3}x - \frac{1}{3}y$. Since, $T_{p'}$ is not our legitimate translation, 60° rotation about p is not our symmetry. On the other hand, the rotation by 120° around the point p is $T_p C_6^2 T_p^{-1} = T_{p'} C_6^2$ where $p' = x$. Therefore it belongs to our symmetry. Similarly the reflection about the line passing through p and y is written as $T_p \sigma_{-x+2y} T_p^{-1} = T_{p'} \sigma_{-x+2y}$ with $p' = x$.

Therefore any operator representing the symmetry operation can be written as a product of some combination of T_x, T_y, C_6, σ_x. Thus, T_x, $T_t = T_{x+y} = T_x T_y$, C_6, σ_x are generators of our symmetry group.

It is easy to verify the following commutation relations between the generators:

$$\sigma_x C_6 = C_6^5 \sigma_x, \tag{12}$$
$$\sigma_x T_t = T_x^3 T_t^{N-1} \sigma_x, \tag{13}$$
$$\sigma_x T_x = T_x \sigma_x, \tag{14}$$
$$C_6 T_t = T_x^{3N-3} T_t^2 C_6, \tag{15}$$
$$C_6 T_x = T_x^{3N-1} T_t C_6, \tag{16}$$
$$T_t T_x = T_x T_t. \tag{17}$$

Using the four generators T_x, T_t, C_6, σ_x with the above commutation relations, our symmetry group is written as

$$G = \{T_x^m T_t^n C_6^p \sigma_x^q \mid m = 0, \cdots, 3N - 1; n = 0, \cdots, N - 1; p = 0, \cdots, 5; q = 0, 1\}, \tag{18}$$

where $T_x^{3N} = T_t^N = C_6^6 = \sigma_x^2 = 1$. This group is exactly identical to the wallpaper group, p6mm.

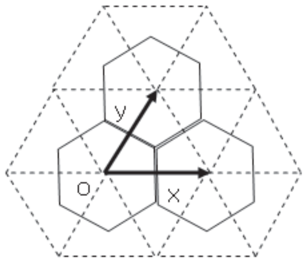

Fig. 3. The solid and dotted lines represent the honeycomb and triangle lattice systems, respectively. Note that the center of each triangle corresponds to a vertex of some hexagon and vice versa.

2.2 Honeycomb

We now consider the honeycomb lattice system. As shown in Figure 3, we first note that in the triangle lattice the collection of the centers of all the triangles forms the honeycomb lattice. We use the same coordinates using the two unit vectors x and y as in the triangle lattice shown in Figure 1. Obviously here C_6 and σ_x are symmetries of our honeycomb lattice too. We show that these two symmetries with translations can generate all the symmetries inherent in the honeycomb lattice. For example in Figure 4, the reflection σ_t about the line passing through the origin and $t = x+y$ can be written as $\sigma_t = C_6\sigma_x$. The symmetries about any vertex of a hexagon in the lattice can be also shown to be generated by C_6, σ_x and translations. For example, let $w = -\frac{1}{3}x+\frac{2}{3}y$ and the $60°$ rotation about this point be $C_{6,w}$. Then $C_{6,w} = T_w^{-1}C_6T_w = T_w^{-1}T_yC_6$, which is not a member of our symmetry. However, the symmetry $C_{6,w}^2$ can be shown to be $C_{6,w}^2 = T_yC_6^2$. The reflection $\sigma_{t'}$ about the line passing through w and $t+w$ can be written as

$$\sigma_{t'} = T_w\sigma_tT_w^{-1} = T_x^{-1}T_y\sigma_t = T_x^{-1}T_yC_6\sigma_x, \tag{19}$$

as shown in Figure 4.

Therefore we conclude that $\{T_x, T_y, C_6, \sigma_x\}$ are the generators of our symmetry group in the honeycomb lattice. We find that the symmetry group for the honeycomb lattice is the same as that for the triangle lattice. To see this more easily, we note that whenever all the vertices go to vertices in an isometric fashion, then all the centers go to centers in the same way.

We assume the same boundary condition for the honeycomb lattice as that for the triangle lattice (see Fig. 1). Each block has $6(1+3+5+\cdots) = 6\sum_{k=1}^{N}(2k-1) = 6N^2$ triangles or centers of triangles which are the sites of our honeycomb lattice.

3 Irreducible representations

The irreducible characters of our group are calculated by using the induced representation method. We find that the group $G = \langle T_x, T_t, C_6, \sigma_x\rangle$ has the order $36N^2$. It is straightforward to obtain all the irreducible characters of

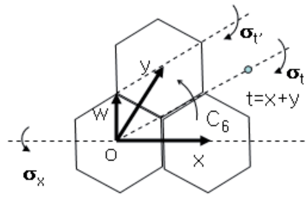

Fig. 4. The rotation C_6 and the reflection σ_x are presented. The reflection σ_t about the line passing through the origin and $t = x+y$, and the reflection $\sigma_{t'}$ about the line passing through w and $t+w$ are shown.

G. The subgroup generated by C_6, σ_x is isomorphic to the *dihedral group* D_6, and the subgroup generated by T_x and T_t is isomorphic to $\mathbf{Z}_{3N} \times \mathbf{Z}_N$. Then, G is isomorphic to a semidirect product of D_6 by an abelian group $\mathbf{Z}_{3N} \times \mathbf{Z}_N$. By applying the standard method of little group in representation theory of finite groups [11], we get all the irreducible characters of G.

The following is the list of all the distinct irreducible characters of G. Here the indices a and b used below play the role of the wave number in the representation of space group.

Case 1. $a = b = 0$
$\Psi_{a,b,k,l}$: 1-dimensional characters $(k, l = 0, 1)$

$$\Psi_{a,b,k,l}(T_x^mT_t^nC_6^p\sigma_x^q) = (-1)^{kp+lq} \tag{20}$$

$\Psi_{a,b,h}$: 2-dimensional characters with $h = 1, 2$

$$\Psi_{a,b,h}(T_x^mT_t^nC_6^p\sigma_x^q) = \begin{cases} 2\cos\frac{\pi hp}{3} & \text{for } q = 0 \\ 0 & \text{for } q = 1 \end{cases} \tag{21}$$

Case 2. $a = N$, $b = 0$
$\Psi_{a,b,l}$: 2-dimensional characters $(l = 0, 1)$

$$\Psi_{a,b,l}(T_x^mT_t^nC_6^p\sigma_x^q) = \begin{cases} 2\cos\frac{2\pi m}{3}(-1)^{lq} & \text{for } p \text{ even} \\ 0 & \text{for } p \text{ odd} \end{cases} \tag{22}$$

$\Psi_{a,b}$: 4-dimensional characters

$$\Psi_{a,b}(T_x^mT_t^nC_6^p\sigma_x^q) = \begin{cases} 4\cos\frac{\pi p}{3}\cos\frac{2\pi m}{3} & \text{for p even and } q = 0 \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

Case 3. $a = 0$, $b = \frac{N}{2}$ (N even)
$\Psi_{a,b,k,l}$: 3-dimensional characters $(k, l = 0, 1)$

$$\Psi_{a,b,k,l}(T_x^mT_t^nC_6^p\sigma_x^q) = \begin{cases} (-1)^{k\frac{p}{3}}[(-1)^n + (-1)^{m+n} + (-1)^{m+2n}] & \text{for } p = 0, 3,\ q = 0 \\ (-1)^{n+k\frac{p}{3}+l} & \text{for } p = 0, 3,\ q = 1 \\ (-1)^{m+2n+k\frac{p-4}{3}+l} & \text{for } p = 1, 4,\ q = 1 \\ (-1)^{m+n+k\frac{p-2}{3}+l} & \text{for } p = 2, 5,\ q = 1 \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

Case 4. $a = 0$, $1 \leq b \leq [\frac{N-1}{2}]$
$\Psi_{a,b,k}$: 6-dimensional characters ($k = 0, 1$)

$$\Psi_{a,b,k}(T_x^m T_t^n C_6^p \sigma_x^q) =$$
$$\begin{cases} 2\cos(\frac{2\pi}{N}bn) + 2\cos(\frac{2\pi}{N}b(m+n)) \\ \quad + 2\cos(\frac{2\pi}{N}b(m+2n)) & \text{for } p=0,\ q=0 \\ 2(-1)^k\cos(\frac{2\pi}{N}b(m+2n)) & \text{for } p=1,\ q=1 \\ 2(-1)^k\cos(\frac{2\pi}{N}bn) & \text{for } p=3,\ q=1 \\ 2(-1)^k\cos(\frac{2\pi}{N}b(m+n)) & \text{for } p=5,\ q=1 \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

Case 5. $1 \leq a \leq [\frac{3N-1}{2}]$, $a \neq N$, $b = 0$
$\Psi_{a,b,k}$: 6-dimensional characters ($k = 0, 1$)

$$\Psi_{a,b,k}(T_x^m T_t^n C_6^p \sigma_x^q) =$$
$$\begin{cases} 2\cos(\frac{2\pi}{3N}am) + 2\cos(\frac{2\pi}{3N}a(m+3n)) \\ \quad + 2\cos(\frac{2\pi}{3N}a(2m+3n)) & \text{for } p=0,\ q=0 \\ 2(-1)^k\cos(\frac{2\pi}{3N}a(2m+3n)) & \text{for } p=0,\ q=1 \\ 2(-1)^k\cos(\frac{2\pi}{3N}a(m+3n)) & \text{for } p=2,\ q=1 \\ 2(-1)^k\cos(\frac{2\pi}{3N}am) & \text{for } p=4,\ q=1 \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

Case 6. $1 \leq a \leq 2b - N - 1$, $[\frac{N+1}{2}] + 1 \leq b \leq N - 1$
$\Psi_{a,b}$: 12-dimensional characters

$$\Psi_{a,b}(T_x^m T_t^n C_6^p \sigma_x^q) =$$
$$\begin{cases} 2\cos[\frac{2\pi}{3N}(am+3bn)] \\ \quad + 2\cos[\frac{2\pi}{3N}((2a-3b)m+3(a-b)n)] \\ \quad + 2\cos[\frac{2\pi}{3N}((a-3b)m+3(a-2b)n)] \\ \quad + 2\cos[\frac{2\pi}{3N}(am+3(a-b)n)] \\ \quad + 2\cos[\frac{2\pi}{3N}((a-3b)m-3bn)] \\ \quad + 2\cos[\frac{2\pi}{3N}((2a-3b)m+3(a-2b)n)] \\ & \text{for } p=0,\ q=0 \\ 0 & \text{otherwise} \end{cases} \tag{27}$$

4 Reducible representation of the Fock space

For a given vector space, in order to calculate a character $\chi(g)$ for an element g of a group G, we set up an orthonormal basis $\{|\psi_i\rangle | i \in I\}$ for the vector space. Then the character is given by

$$\chi(g) = \sum_{i \in I} \langle \psi_i | g | \psi_i \rangle. \tag{28}$$

The relevant group G we consider here has the generators $\langle T_x, T_t, C_6, \sigma_x \rangle$. We can describe the lattice sites as $ix + jy$ with two integers i, j for the triangle lattice, and $ix + jy \pm w$ for the honeycomb lattice. The vector space here is the Fock space, where we denote the fundamental creation operators defined in the triangle and the honeycomb lattice as $c^\dagger_{ix+jy}$ and $c^\dagger_{ix+jy\pm w}$, respectively.

4.1 Triangle

For the translation, as we see in Figure 1, it is assumed that the creation operators satisfy the periodic boundary condition

$$c^\dagger_{ix+jy} = c^\dagger_{(i+3N)x+jy} = c^\dagger_{ix+(j+3N)y} = c^\dagger_{(i+N)x+(j+N)y}. \tag{29}$$

The elementary symmetry operators are related to the creation operators as follows:

$$T_x c^\dagger_{ix+jy} = c^\dagger_{(i+1)x+jy} T_x, \tag{30}$$
$$T_y c^\dagger_{ix+jy} = c^\dagger_{ix+(j+1)y} T_y, \tag{31}$$
$$C_6 c^\dagger_{ix+jy} = c^\dagger_{-jx+(i+j)y} C_6, \tag{32}$$
$$\sigma_x c^\dagger_{ix+jy} = c^\dagger_{(i+j)x-jy} \sigma_x, \tag{33}$$
$$|0\rangle = T_x|0\rangle = T_y|0\rangle = C_6|0\rangle = \sigma_x|0\rangle. \tag{34}$$

From these relations between creation operators and symmetry operators, we determine how the operators act on the M fermion Fock space.

We note that one (hexagon-looking) block of the $3N^2$ lattice sites in Figure 1 is equivalent to the parallelogram of the $3N \times N$ sites $\{ix + jt | i = 0, \cdots, 3N-1; j = 0, \cdots, N-1\}$ where $t = x + y$ by using the boundary condition.

The Fock space $\mathcal{F}$ consists of M fermions living in $3N \times N$ sites. We let $c^\dagger_{ix+jt} \equiv c^\dagger_{i\ j}$, and find the basis of the Fock space as

$$\mathcal{F} = \mathrm{Span}\{c^\dagger_{i_1\ j_1} \cdots c^\dagger_{i_M\ j_M}|0\rangle | i_1 N + j_1 < \cdots < i_M N + j_M\}, \tag{35}$$

where $0 \leq i_k \leq 3N-1$, and $0 \leq j_k \leq N-1$.

For the calculation of the character $\chi(g)$, we first consider so called g-orbits which are written as

$$\mathcal{O}_{i\ j} \equiv \{g^l(c^\dagger_{i\ j}) | l = 0, 1, 2, 3, \cdots\}, \tag{36}$$

where $g(c^\dagger_{i\ j}) \equiv c^\dagger_{i'\ j'}$ which satisfies $g c^\dagger_{i\ j} = c^\dagger_{i'\ j'} g$ from the given commutation relations above between group elements and creation operators. As we emphasize in the previous paper [8], the key observation is that

$$\langle 0 | c_{i_M\ j_M} \cdots c_{i_1\ j_1} | g | c^\dagger_{i_1\ j_1} \cdots c^\dagger_{i_M\ j_M} | 0 \rangle = \pm 1 \tag{37}$$

holds if and only if the set, $\{c^\dagger_{i_1\ j_1}, \cdots, c^\dagger_{i_M\ j_M}\}$, is a disjoint union of the g-orbits, $\mathcal{O}_{i_1\ j_1} \cup \cdots \cup \mathcal{O}_{i_M\ j_M}$. The main point here is that given $g \in G$ the character $\chi(g)$ is determined by the number of its orbits and their sizes.

In order to find $\chi(g)$, we now analyze the orbits generated by $g = T_x^m T_t^n C_6^p \sigma_x^q \in G$ in the $3N \times N$ sites in the triangle lattice.

Let us present here a typical procedure to find $\chi(g)$ as an example for the case of, say, $q=1$, $p=1$. Let $g = T_x^m T_t^n C_6 \sigma_x$. Then,

$$g(c^\dagger_{i\ j}) = c^\dagger_{-i+m,\ i+j+n}, g^2(c^\dagger_{i\ j}) = c^\dagger_{i,\ j+m+2n},$$
$$g^3(c^\dagger_{i\ j}) = c^\dagger_{-i+m,\ i+j+m+3n}, g^4(c^\dagger_{i\ j}) = c^\dagger_{i,\ j+2(m+2n)},$$
$$g^5(c^\dagger_{i\ j}) = c^\dagger_{-i+m,\ i+j+2m+5n}, g^6(c^\dagger_{i\ j}) = c^\dagger_{i,\ j+3(m+2n)}.$$

Let $e \equiv m+2n$, $0 \le e < N$, $\frac{e}{N} = \frac{r}{s}$, $\gcd(r,s)=1$. If s is even, $g^{2s}(c^\dagger_{i\ j}) = c^\dagger_{i\ j}$ and we have $\frac{3N^2}{2s}$ orbits of size $2s$ and the trace is $(-1)^{\frac{M}{2s}}\ {}_{\frac{3N^2}{2s}}\mathrm{C}_{\frac{M}{2s}}$ if $v=0$ and the trace is 0 if $v=1$ where $\frac{M}{s} = 2u+v$. If s is odd, let $s = 2l+1$ $(l = 0,1,\cdots)$. Then $g^s(c^\dagger_{i\ j}) = c^\dagger_{i\ j}$ for those (i,j) which satisfy the equations $i = -i+m+3\alpha N$ and $j = i+j+lm+(2l+1)n+\beta N$ for some integer α and β. As a result, we find that there is just one $i_0 = (m+3\alpha N)/2 = -\beta N - lm - (2l+1)n$, from which we obtain $(2l+1)(m+2n) = N(-2\beta-3\alpha)$ as a consistency equation. Therefore, there are N solutions, (i_0, j) where j runs from 0 to $N-1$. Furthermore, $g^{2s}(c^\dagger_{i\ j}) = c^\dagger_{i\ j}$ for any (i,j). Therefore there are $\frac{N}{s}$ orbits of size s and $\frac{N(3N-1)}{2s}$ orbits of size $2s$. The trace is then

$$\sum_{k=0}^{u} (-1)^{(u-k)}\ {}_{\frac{N(3N-1)}{2s}}\mathrm{C}_{u-k}\ {}_{\frac{N}{s}}\mathrm{C}_{2k+v}. \tag{38}$$

For other cases of q and p, the procedures are not different from the above case. We present here the summary of the characters of the reducible representation of our Fock space with M-fermions. As usual, we use the convention that ${}_A\mathrm{C}_B = 0$ if $A<B$ or $B<0$.

Case 1. $q=0, p=0$
$\frac{m}{3N} = \frac{r_1}{s_1}$, $\frac{n}{N} = \frac{r_2}{s_2}$, $\gcd(r_1,s_1) = \gcd(r_2,s_2) = 1, s = \mathrm{lcm}(s_1,s_2)$

$$\chi(T_x^m T_t^n) = \begin{cases} (-1)^{\frac{M}{s}(s-1)}\ {}_{\frac{3N^2}{s}}\mathrm{C}_{\frac{M}{s}} & \text{if } s|M \\ 0 & \text{otherwise} \end{cases} \tag{39}$$

Case 2. $q=0, p=1,5$
$f = N \bmod 2$, $M = 6t+r$, $0 \le r \le 5$.

$$\chi(T_x^m T_t^n C_6) = \chi(T_x^m T_t^n C_6^5) =$$
$$\begin{cases} (-1)^t\ {}_{\frac{N^2}{2}}\mathrm{C}_t & \text{if } f=0,\ r=0 \\ (-1)^t\ {}_{\frac{N^2-2}{2}}\mathrm{C}_t & \text{if } f=0,\ r=1,4 \\ 0 & \text{if } f=0,\ r=3 \\ (-1)^{t+1}\ {}_{\frac{N^2-2}{2}}\mathrm{C}_t & \text{if } f=0,\ r=2,5 \\ (-1)^t\ {}_{\frac{N^2-1}{2}}\mathrm{C}_t & \text{if } f=1,\ r=0,1 \\ (-1)^{t+1}\ {}_{\frac{N^2-1}{2}}\mathrm{C}_t & \text{if } f=1,\ r=2,3 \\ 0 & \text{if } f=1,\ r=4,5 \end{cases} \tag{40}$$

Case 3. $q=0, p=2,4$
$M = 3t+r$, $0 \le r \le 2$.

$$\chi(T_x^m T_t^n C_6^2) = \chi(T_x^m T_t^n C_6^4) =$$
$$\begin{cases} {}_{N^2}\mathrm{C}_t & \text{if } r=0 \\ 3\ {}_{N^2-1}\mathrm{C}_t & \text{if } r=1,2 \text{ and if } 3|m \\ 0 & \text{otherwise} \end{cases} \tag{41}$$

Case 4. $q=0, p=3$
$f = N \bmod 2$, $g = m \bmod 2$, $h = n \bmod 2$, $M = 2t + r, r = 0,1$.

$$\chi(T_x^m T_t^n C_4^3) =$$
$$\begin{cases} (-1)^t\ {}_{\frac{3N^2-4}{2}}\mathrm{C}_t + 6(-1)^{t-1}\ {}_{\frac{3N^2-4}{2}}\mathrm{C}_{t-1} + (-1)^{t-2}\ {}_{\frac{3N^2-4}{2}}\mathrm{C}_{t-2} & \text{if } f=g=h=r=0 \\ 4(-1)^t\ {}_{\frac{3N^2-4}{2}}\mathrm{C}_t + 4(-1)^{t-1}\ {}_{\frac{3N^2-4}{2}}\mathrm{C}_{t-1} & \text{if } f=g=h=0, r=1 \\ (-1)^t\ {}_{\frac{3N^2}{2}}\mathrm{C}_t & \text{if } f=0,\ g \text{ or } h=1,\ r=0 \\ 0 & \text{if } f=0,\ g \text{ or } h=1,\ r=1 \\ (-1)^t\ {}_{\frac{3N^2-1}{2}}\mathrm{C}_t & \text{if } f=1 \end{cases} \tag{42}$$

Case 5. $q=1, p=0$
$2m+3n = e$, $\frac{e}{3N} = \frac{r}{s}$, $\gcd(r,s)=1$, $h = s \bmod 2$, $\frac{M}{s} = 2u+v$.

$$\chi(T_x^m T_t^n \sigma) =$$
$$\begin{cases} (-1)^{\frac{M}{2s}}\ {}_{\frac{3N^2}{2s}}\mathrm{C}_{\frac{M}{2s}} & \text{if } h=0,\ v=0 \\ \sum_{k=0}^{u} (-1)^{(u-k)}\ {}_{\frac{3N(N-1)}{2s}}\mathrm{C}_{u-k}\ {}_{\frac{3N}{s}}\mathrm{C}_{2k+v} & \text{if } h=1 \\ 0 & \text{otherwise} \end{cases} \tag{43}$$

Case 6. $q=1, p=1$
$m+2n = e$, $\frac{e}{N} = \frac{r}{s}$, $\gcd(r,s)=1$, $h = s \bmod 2$, $\frac{M}{s} = 2u+v$.

$$\chi(T_x^m T_t^n C_6 \sigma) =$$
$$\begin{cases} (-1)^{\frac{M}{2s}}\ {}_{\frac{3N^2}{2s}}\mathrm{C}_{\frac{M}{2s}} & \text{if } h=0,\ v=0 \\ \sum_{k=0}^{u} (-1)^{(u-k)}\ {}_{\frac{N(3N-1)}{2s}}\mathrm{C}_{u-k}\ {}_{\frac{N}{s}}\mathrm{C}_{2k+v} & \text{if } h=1 \\ 0 & \text{otherwise} \end{cases} \tag{44}$$

Case 7. $q=1, p=2$
$m+3n = e$, $\frac{e}{3N} = \frac{r}{s}$, $\gcd(r,s)=1$, $h = s \bmod 2$, $\frac{M}{s} = 2u+v$.

$$\chi(T_x^m T_t^n C_6^2 \sigma) =$$
$$\begin{cases} (-1)^{\frac{M}{2s}}\ {}_{\frac{3N^2}{2s}}\mathrm{C}_{\frac{M}{2s}} & \text{if } h=0,\ v=0 \\ \sum_{k=0}^{u} (-1)^{(u-k)}\ {}_{\frac{3N(N-1)}{2s}}\mathrm{C}_{u-k}\ {}_{\frac{3N}{s}}\mathrm{C}_{2k+v} & \text{if } h=1 \\ 0 & \text{otherwise} \end{cases} \tag{45}$$

Case 8. $q=1, p=3$
$n = e$, $\frac{e}{N} = \frac{r}{s}$, $\gcd(r,s)=1$, $h = s \bmod 2$, $\frac{M}{s} = 2u+v$.

$$\chi(T_x^m T_t^n C_6^3 \sigma) =$$
$$\begin{cases} (-1)^{\frac{M}{2s}}\ {}_{\frac{3N^2}{2s}}\mathrm{C}_{\frac{M}{2s}} & \text{if } h=0,\ v=0 \\ \sum_{k=0}^{u} (-1)^{(u-k)}\ {}_{\frac{N(3N-1)}{2s}}\mathrm{C}_{u-k}\ {}_{\frac{N}{s}}\mathrm{C}_{2k+v} & \text{if } h=1 \\ 0 & \text{otherwise} \end{cases} \tag{46}$$

Case 9. $q = 1$, $p = 4$
$m = e$, $\frac{e}{3N} = \frac{r}{s}$, $\gcd(r, s) = 1$, $h = s \bmod 2$,
$\frac{M}{s} = 2u + v$.

$$\chi(T_x^m T_t^n C_6^4 \sigma) = \begin{cases} (-1)^{\frac{M}{2s}} {}_{\frac{3N^2}{2s}}\mathrm{C}_{\frac{M}{2s}} & \text{if } h = 0,\ v = 0 \\ \sum_{k=0}^{u} (-1)^{(u-k)}_{\frac{3N(N-1)}{2s}} \mathrm{C}_{u-k \frac{3N}{s}} \mathrm{C}_{2k+v} & \text{if } h = 1 \\ 0 & \text{otherwise} \end{cases} \quad (47)$$

Case 10. $q = 1$, $p = 5$
$m + n = e$, $\frac{e}{N} = \frac{r}{s}$, $\gcd(r, s) = 1$, $h = s \bmod 2$,
$\frac{M}{s} = 2u + v$.

$$\chi(T_x^m T_t^n C_6^5 \sigma) = \begin{cases} (-1)^{\frac{M}{2s}} {}_{\frac{3N^2}{2s}}\mathrm{C}_{\frac{M}{2s}} & \text{if } h = 0,\ v = 0 \\ \sum_{k=0}^{u} (-1)^{(u-k)}_{\frac{N(3N-1)}{2s}} \mathrm{C}_{u-k \frac{N}{s}} \mathrm{C}_{2k+v} & \text{if } h = 1 \\ 0 & \text{otherwise} \end{cases} \quad (48)$$

4.2 Honeycomb

We have to use three integers when we describe creation operators for the Fock space in the honeycomb lattice system, where the number of sites is given by $3N \times N \times 2$. As in the previous section, we use three vectors x, $t = x + y$ and $w = -\frac{1}{3}x + \frac{2}{3}y$, and we let $c^\dagger_{ix+jt+(2k-1)w} \equiv c^\dagger_{i\ j\ k}$ ($k = 0, 1$). Then, we find the basis of the Fock space $\mathcal{F}$ with M fermions as

$$\mathcal{F} = \mathrm{Span}\{c^\dagger_{i_1\ j_1\ k_1} \cdots c^\dagger_{i_M\ j_M\ k_M}|0\rangle | i_1 2N + j_1 2 + k_1 < \cdots < i_M 2N + j_M 2 + k_M\} \quad (49)$$

where $0 \le i_l \le 3N-1$, and $0 \le j_l \le N-1$, and $0 \le k_l \le 1$.

Here the idea using orbits is the same as the case of triangle when we calculate the character $\chi(g)$. Instead of analytic formula, we do numerical calculation for the orbits using the computer. The numerical results are used when we find multiplicities in the next section.

5 Multiplicity calculation

For the group $G = \langle T_x, T_t, C_6, \sigma_x \rangle$, we have found the irreducible representations and the reducible characters of the Fock space. Using these characters, we calculate multiplicities for the corresponding irreducible sectors.

We simply present the multiplicities of the six cases of the irreducible representations which were given in the previous section. We denote the multiplicity of a given irreducible representation Ψ by $\mu(\Psi)$.

Example 1: Triangle lattice system with $N = 4$, $M = 8$.

Case 1. $a = b = 0$
$\Psi_{a,b,k,l}$: 1-dimensional characters ($k, l = 0, 1$)

$$\mu(\Psi_{0,0,0,0}) = 655906, \mu(\Psi_{0,0,1,0}) = 654851, \\ \mu(\Psi_{0,0,0,1}) = 655742, \mu(\Psi_{0,0,1,1}) = 654277.$$

$\Psi_{a,b,h}$: 2-dimensional characters with $h = 1, 2$

$$\mu(\Psi_{0,0,1}) = 1309079, \mu(\Psi_{0,0,2}) = 1311592.$$

Case 2. $a = N$, $b = 0$
$\Psi_{a,b,l}$: 2-dimensional characters ($l = 0, 1$)

$$\mu(\Psi_{4,0,0}) = 1310757, \mu(\Psi_{4,0,1}) = 1310019.$$

$\Psi_{a,b}$: 4-dimensional characters

$$\mu(\Psi_{4,0}) = 2620671.$$

Case 3. $a = 0$, $b = \frac{N}{2}$ (N even)
$\Psi_{a,b,k,l}$: 3-dimensional characters ($k, l = 0, 1$)

$$\mu(\Psi_{0,2,0,0}) = 1964838, \mu(\Psi_{0,2,1,0}) = 1966580, \\ \mu(\Psi_{0,2,0,1}) = 1964670, \mu(\Psi_{0,2,1,1}) = 1966008.$$

Case 4. $a = 0$, $1 \le b \le [\frac{N-1}{2}]$
$\Psi_{a,b,k}$: 6-dimensional characters ($k = 0, 1$)

$$\mu(\Psi_{0,1,0}) = 3930388, \mu(\Psi_{0,1,1}) = 3930828.$$

Case 5. $1 \le a \le [\frac{3N-1}{2}]$, $a \ne N$, $b = 0$
$\Psi_{a,b,k}$: 6-dimensional characters ($k = 0, 1$)

$$\mu(\Psi_{1,0,0}) = 3930972, \mu(\Psi_{1,0,1}) = 3930244, \\ \mu(\Psi_{2,0,0}) = 3931418, \mu(\Psi_{2,0,1}) = 3930678, \\ \mu(\Psi_{3,0,0}) = 3930972, \mu(\Psi_{3,0,1}) = 3930244, \\ \mu(\Psi_{5,0,0}) = 3930972, \mu(\Psi_{5,0,1}) = 3930244.$$

Case 6. $1 \le a \le 2b - N - 1$, $[\frac{N+1}{2}] + 1 \le b \le N - 1$
$\Psi_{a,b}$: 12-dimensional characters

$$\mu(\Psi_{1,3}) = 7861216.$$

To check the consistency, the following equation must hold:

$${}_{3N^2}\mathrm{C}_M = \sum_{\Psi} \dim(\Psi) \times \mu(\Psi). \quad (50)$$

We note the consistency with equation (50):

$$\begin{aligned} {}_{3\cdot 4^2}\mathrm{C}_8 &= 377348994 \\ &= \ 1 \times (655906 + 654851 + 655742 + 654277) \\ &\quad +2 \times (1309079 + 1311592) \\ &\quad +2 \times (1310757 + 1310019) \\ &\quad +4 \times (2620671) \\ &\quad +3 \times (1964838 + 1966580 + 1964670 + 1966008) \\ &\quad +6 \times (3930388 + 3930828) \\ &\quad +6 \times (3930972 + 3930244 + 3931418 + 3930678 \\ &\qquad + 3930972 + 3930244 + 3930972 + 3930244) \\ &\quad +12 \times (7861216). \end{aligned} \quad (51)$$

Example 2: Honeycomb lattice system with $N = 4$, $M = 8$.

Case 1. $a = b = 0$

$\Psi_{a,b,k,l}$: 1-dimensional characters $(k,l=0,1)$

$$\mu(\Psi_{0,0,0,0}) = 230220931,\ \mu(\Psi_{0,0,1,0}) = 230209731,$$
$$\mu(\Psi_{0,0,0,1}) = 230234043,\ \mu(\Psi_{0,0,1,1}) = 230180383.$$

$\Psi_{a,b,h}$: 2-dimensional characters with $h=1,2$

$$\mu(\Psi_{0,0,1}) = 460389649,\ \mu(\Psi_{0,0,2}) = 460454509.$$

Case 2. $a=N$, $b=0$
$\Psi_{a,b,l}$: 2-dimensional characters $(l=0,1)$

$$\mu(\Psi_{4,0,0}) = 460430197,\ \mu(\Psi_{4,0,1}) = 460413961.$$

$\Psi_{a,b}$: 4-dimensional characters

$$\mu(\Psi_{4,0}) = 920844623.$$

Case 3. $a=0$, $b=\frac{N}{2}$ (N even)
$\Psi_{a,b,k,l}$: 3-dimensional characters $(k,l=0,1)$

$$\mu(\Psi_{0,2,0,0}) = 690626776,\ \mu(\Psi_{0,2,1,0}) = 690648000,$$
$$\mu(\Psi_{0,2,0,1}) = 690639880,\ \mu(\Psi_{0,2,1,1}) = 690618656.$$

Case 4. $a=0$, $1 \le b \le [\frac{N-1}{2}]$
$\Psi_{a,b,k}$: 6-dimensional characters $(k=0,1)$

$$\mu(\Psi_{0,1,0}) = 1381237264,\ \mu(\Psi_{0,1,1}) = 1381279856.$$

Case 5. $1 \le a \le [\frac{3N-1}{2}]$, $a \ne N$, $b=0$
$\Psi_{a,b,k}$: 6-dimensional characters $(k=0,1)$

$$\mu(\Psi_{1,0,0}) = 1381266656,\ \mu(\Psi_{1,0,1}) = 1381250464,$$
$$\mu(\Psi_{2,0,0}) = 1381274776,\ \mu(\Psi_{2,0,1}) = 1381258536,$$
$$\mu(\Psi_{3,0,0}) = 1381266656,\ \mu(\Psi_{3,0,1}) = 1381250464,$$
$$\mu(\Psi_{5,0,0}) = 1381266656,\ \mu(\Psi_{5,0,1}) = 1381250464.$$

Case 6. $1 \le a \le 2b-N-1$, $[\frac{N+1}{2}]+1 \le b \le N-1$
$\Psi_{a,b}$: 12-dimensional characters

$$\mu(\Psi_{1,3}) = 2762517120.$$

We should also check the consistency:

$$\begin{aligned}
{}_{6\cdot 4^2}\mathrm{C}_8 &= 132601016340 \\
&= \ 1 \times (230220931 + 230209731 \\
&\qquad + 230234043 + 230180383) \\
&\quad +2 \times (460389649 + 460454509) \\
&\quad +2 \times (460430197 + 460413961) \\
&\quad +4 \times (920844623) \\
&\quad +3 \times (690626776 + 690648000 \\
&\qquad + 690639880 + 690618656) \\
&\quad +6 \times (1381237264 + 1381279856) \\
&\quad +6 \times (1381266656 + 1381250464 \\
&\qquad + 1381274776 + 1381258536 \\
&\qquad + 1381266656 + 1381250464 \\
&\qquad + 1381266656 + 1381250464) \\
&\quad +12 \times (2762517120).
\end{aligned} \tag{52}$$

This multiplicity is the dimension of the Hamiltonian submatrix, which should be diagonalized by some method beyond group theory in exact diagonalization. We emphasize that the structure of the quantum numbers for the honeycomb lattice system is exactly the same as that for the triangle lattice system.

6 Conclusion

We have considered a many particle system related with the plane group, p6mm. The corresponding symmetry group is isomorphic to a semidirect product of dihedral group D_6 and an abelian group $\mathbf{Z}_{3N} \times \mathbf{Z}_N$. By finding the irreducible representations of the group, we can assign quantum numbers to a many particle system. We also find the reducible characters for the M-fermion Fock space. Using the irreducible representations and the reducible characters, we find multiplicities by calculating the inner product between irreducible and reducible characters. We check the consistency of the dimensions.

This basic approach to symmetries in this paper will be useful in many exact diagonalization studies [12–17]. This work will be useful to study graphene, where the fractional quantum Hall effect is observed [18]. Our work can be extended to the case of three-dimensional lattice systems. It will not be difficult to find multiplicities of the Fock space for three-dimensional lattice systems.

This work was done when one of the authors (M.H.C.) visited to KIAS.

References

1. R. Maezono, A. Ma, M.D. Towler, R.J. Needs, Phys. Rev. Lett. **98**, 025701 (2007)
2. N. Laflorencie, D. Poilblanc, Lect. Notes Phys. **645**, 227 (2004)
3. U. Schollwöck, Rev. Mod. Phys. **77**, 259 (2005)
4. D. Schattschneider, Am. Math. Month. **85**, 439 (1978)
5. Th. Hahn, *International Tables for Crystallography: Brief Teaching Edition of Volume A, Space Group Symmetry* (Springer, Dordrecht, 2005)
6. O.V. Kovalev, *Irreducible Representations of Space Groups* (Gordon and Breach, New York, 1965)
7. M. Hammermesh, *Group Theory* (Addison-Wesley, New York, 1962)
8. B. Kim, M.H. Chung, J.H. Kwon, Lett. Math. Phys. **78**, 73 (2006)
9. W.J. He, T. Cui, Y.M. Ma, Z.M. Liu, G.T. Zou, Phys. Rev. B **68**, 195104 (2003)
10. J.C. Meyer, A.K. Geim, M.I. Katsnelson, K.S. Novoselov, T.J. Booth, S. Roth, Nature **446**, 60 (2007)
11. J.P. Serre, *Linear representations of finite groups* (Springer-Verlag, Berlin, 1977)
12. O. Golinelli, Th. Jolicoeur, R. Lacaze, Phys. Rev. B **50**, 3037 (1994)
13. F.D.M. Haldane, Phys. Rev. Lett. **55**, 2095 (1985)
14. M.H. Chung, J. Hong, J.H. Kwon, Phys. Rev. B **55**, 2249 (1997)
15. C.T. Shih, T.K. Lee, E. Eder, C.Y. Mou, Y.C. Chen, Phys. Rev. Lett. **92**, 227002 (2004)
16. D. Poilblanc, D.J. Scalapino, S. Capponi, Phys. Rev. Lett. **91**, 137203 (2003)
17. E. Dagotto, Rev. Mod. Phys. **66**, 763 (1994)
18. Z. Jiang, Y. Zhang, H.L. Stormer, P. Kim, Phys. Rev. Lett. **99**, 106802 (2007)